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Automorphism groups and isomorphisms of Cayley digraphs[1]

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Abstract

We call a Cayley digraph $X = \mathrm{Cay}(G,S)$ *normal* for G if the right regular representation $R(G)$ of G is normal in the full automorphism group $\mathrm{Aut}(X)$ of X. In this paper we give some examples of normal and nonnormal Cayley digraphs and survey some results about the normality of Cayley digraphs. We also propose some conjectures and problems about them. In the last section of this paper we discuss a problem about isomorphisms of Cayley digraphs.

Keywords: Cayley digraphs; Normal Cayley digraphs; CI-subsets of a group

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1. Introduction

Let G be a finite group and S a subset of G not containing the identity element 1. We define the Cayley digraph $X = \mathrm{Cay}(G,S)$ of G with respect to S by

$$V(X) = G,$$
$$E(X) = \{(g, sg) \mid g \in G, s \in S\}.$$

The following obvious facts are basic for Cayley digraphs.

Proposition 1.1. *Let $X = \mathrm{Cay}(G,S)$ be a Cayley digraph of G with respect to S. Then*
(1) $\mathrm{Aut}(X)$ *contains the right regular representation $R(G)$ of G, so X is vertex-transitive.*
(2) *X is connected if and only if $G = \langle S\rangle$.*
(3) *X is undirected if and only if $S^{-1} = S$.*

The following fact is well known.

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Proposition 1.2. *A digraph $X=(V,E)$ is a Cayley digraph of a group if and only if* $\mathrm{Aut}(X)$ *contains a regular subgroup.*

In this paper we are concerned with two problems about Cayley digraphs. One is about the full automorphism group of a Cayley digraph, the other is about isomorphisms between Cayley digraphs. In this introductory section we shall introduce some concepts.

Let $X=\mathrm{Cay}(G,S)$ be a Cayley digraph of G with respect to S, and let

$$\mathrm{Aut}(G,S)=\{\alpha\in\mathrm{Aut}(G)\mid S^{\alpha}=S\}.$$

Obviously, $\mathrm{Aut}(X)\geqslant R(G)\mathrm{Aut}(G,S)$. Let $A=\mathrm{Aut}(X)$. We have

Proposition 1.3. (1) $N_A(R(G))=R(G)\mathrm{Aut}(G,S)$;

(2) $A=R(G)\mathrm{Aut}(G,S)$ *is equivalent to* $R(G)\lhd A$.

Proof. Since the normalizer of $R(G)$ in the symmetric group $\mathrm{Sym}(G)$ is the holomorph of G, that is $R(G)\mathrm{Aut}(G)$, we have

$$N_A(R(G))=R(G)\mathrm{Aut}(G)\cap A=R(G)\mathrm{Aut}(G)\cap R(G)A_1=R(G)(\mathrm{Aut}(G)\cap A_1),$$

where A_1 is the stabilizer of the identity element 1 in A.

Obviously, $\mathrm{Aut}(G)\cap A_1=\mathrm{Aut}(G,S)$. Thus, (1) holds.

(2) is an immediate consequence of (1). □

Definition 1.4. The Cayley digraph $X=\mathrm{Cay}(G,S)$ is called *normal* if $R(G)\lhd A=\mathrm{Aut}(X)$.

So, normal Cayley digraphs are just those which have the smallest possible full automorphism groups. The following obvious result is a direct consequence of the above definition and Proposition 1.3.

Proposition 1.5. *Let $X=\mathrm{Cay}(G,S)$ be a Cayley digraph of G with respect to S, and $A=\mathrm{Aut}(X)$. Let A_1 be the stabilizer of the identity element* 1 *in A. Then X is normal if and only if every element of A_1 is an automorphism of the group G.*

We shall give more detailed discussions about normal Cayley digraphs in Section 2. Next we shall give some concepts about the isomorphism problem for Cayley digraphs.

Let $X=\mathrm{Cay}(G,S)$ be a Cayley digraph of G with respect to S. Let $\alpha\in\mathrm{Aut}(G)$. Then it is easy to see that α is a graph isomorphism from $\mathrm{Cay}(G,S)$ to $\mathrm{Cay}(G,S^{\alpha})$. We call these kinds of isomorphisms between Cayley digraphs of G *Cayley isomorphisms.* The subset S is said to be a CI-subset of G (CI stands for 'Cayley isomorphism') if whenever $\mathrm{Cay}(G,T)$ is isomorphic to $\mathrm{Cay}(G,S)$, there is a Cayley isomorphism between them.

In the literature many papers concerning the isomorphism problem are devoted to determining so-called CI- and DCI-groups; see [3,6,7,9,31,37,38] for references. A group

G is called a DCI- (or CI-)group, if every subset (or every inverse-closed subset) of G is a CI-subset.

However, there are very few groups which are DCI- or CI-groups. Some people have considered the following concepts to shed some light on these concepts.

Let m be a positive integer. A group G is called an m-DCI- (or m-CI-) group, if every subset (or every inverse-closed subset) of G of cardinality at most m is a CI-subset. These two concepts are useful for studying the isomorphism problems for Cayley digraphs of small valency. See [11–14] for references.

Recently, the isomorphism problem has been considered for the restricted class of connected Cayley digraphs. A group G is called a weakly m-DCI- (or weakly m-CI-) group, if every generating subset (or every inverse-closed generating subset) of G of cardinality at most m is a CI-subset. These concepts are more important for studying the isomorphism problem of Cayley digraphs, because every Cayley digraph is just a union of isomorphic connected components. Some results for abelian groups can be found in [28–30]. For general groups, I asked the question: is every minimal generating subset for a finite group a CI-subset? (see [39, Problem 8]). I shall give a more detailed discussion about this question in Section 3.

2. Normal Cayley digraphs

In this section we shall give some examples of normal and nonnormal Cayley digraphs, and also we shall survey some known results about them. Note that the normality of a Cayley digraph depends on the group G. For example, K_4 is a Cayley graph of $G = Z_2 \times Z_2$ and of $G = Z_4$. It is normal for $Z_2 \times Z_2$, but it is nonnormal for Z_4.

Example 2.1. Let $X = K_n$ or nK_1, and $n \geqslant 5$. Then X is nonnormal for any regular subgroup G of S_n and A_n.

Example 2.2. Let $G = Z_p$, p a prime. Then all Cayley digraphs, other than K_p or pK_1, are normal.

Proof. (Galois and Burnside). Assume that $X \neq K_p$ or pK_1. Then $\mathrm{Aut}(X)$ is not 2-transitive. It follows that $\mathrm{Aut}(X) \leqslant AG(1, p)$ is solvable and $R(G) = Z_p$ is normal in $\mathrm{Aut}(X)$. □

In the 1970s much work was done for determining so-called graphical regular representations (GRRs) and digraphical regular representations (DRRs) of finite groups (see [4,5,7] for references). A GRR of a group G is an undirected graph X whose automorphism group is isomorphic to G and acts on the vertex set of X regularly. A DRR is a digraph version of GRR. Obviously we have

Example 2.3. All GRRs and DRRs are normal Cayley graphs and digraphs.

The next result is about the lexicographic product construction. It is easy to see that a Cayley digraph $X = \mathrm{Cay}(G,S)$ is a nontrivial lexicographic product if and only if there is a proper nontrivial subgroup H of G such that $S\backslash H$ is a union of left cosets of H. (Note that a disconnected digraph is also a nontrivial lexicographic product.) It is easy to prove that any GRR or DRR must not be a nontrivial lexicographic product, and it must be connected. However, for normal Cayley graphs and digraphs we have the following two propositions (see [35]).

Proposition 2.4. *Let $X = \mathrm{Cay}(G,S)$ be a normal Cayley graph of a group G. Then we have*

(1) *X is disconnected if and only if*

(i) *$G \cong Z_2^{r+1}$ or $Z_4 \times Z_2^{r-1}$, where $r = 1$ or $r \geqslant 5$;*

(ii) *$H = \langle S \rangle \cong Z_2^r$;*

(iii) *$W = \mathrm{Cay}(H,S)$ is a GRR for H.*

(2) *X is a nontrivial lexicographic product if and only if X or its complement X^c is disconnected.*

Proposition 2.5. *Let $X = \mathrm{Cay}(G,S)$ be a normal Cayley digraph of a group G. Then we have*

(1) *X is disconnected if and only if*

(i) *$H = \langle S \rangle$ is a proper nontrivial abelian subgroup of G and $|G:H| = 2$;*

(ii) *for any $b \in G\backslash H$ and $h \in H$, the order of b divides 4 and $b^{-1}hb = h^{-1}$;*

(iii) *$W = \mathrm{Cay}(H,S)$ is a DRR for H.*

(2) *X is a nontrivial lexicographic product if and only if X or its complement X^c is disconnected.*

The next four examples of nonnormal Cayley digraphs are more interesting. The proofs for the nonnormality of the Cayley digraphs in these examples are easy and are omitted.

Example 2.6. Let $X = \mathrm{Cay}(G,S)$. If $\mathrm{Aut}(X)$ acts primitively on $V(X)$, and is a primitive group of almost simple type, then X is nonnormal for G.

Example 2.7. Let $G = Z_p^2 = \{(i,j) \mid i,j = 0,1,\ldots,p-1\}$ and $S = \{(i,0),(0,j) \mid i,j \in Z_p\}$, where $p > 3$ is an odd prime. Then $X = \mathrm{Cay}(G,S)$ has automorphism group $\mathrm{Aut}(X) = S_p \wr Z_2$ in its product action, which is a primitive group acting on $G = Z_p^2$. The graph X and its complement are nonnormal for G.

Example 2.8. The full automorphism group of the incidence graphs of the doubly transitive Hadamard 2-(11,5,2) design and its complementary design is a semidirect product of PSL(2,11) and Z_2. This group has a regular subgroup isomorphic to D_{22}, and the graphs are nonnormal when they are viewed as Cayley graphs of D_{22}.

Example 2.9. The full automorphism group A of the point-hyperplane incidence graph X of $\mathrm{PG}(n-1,q)$, and its bipartite complement X', is a semidirect product of $\mathrm{P\Gamma L}(n,q)$ and Z_2. Let $p=(q^n-1)/(q-1)$ be a prime. Then A has a regular subgroup D_{2p}, and X and X' are nonnormal for D_{2p}.

Now we may ask a question: Does every finite group have at least one normal Cayley digraph and Cayley graph? The next example shows that this is not true for Cayley (undirected) graphs.

Example 2.10. Let $G=Z_4\times Z_2$ or $G=Q_8\times Z_2^m$, $m\geqslant 0$. Then every Cayley graph of G is nonnormal.

The proof of the nonnormality of Cayley graphs of the first group is just by checking. But for the group $G=Q_8\times Z_2^m$, letting $X=\mathrm{Cay}(G,S)$ be any Cayley graph of G, we may easily check that the mapping $\alpha: g\mapsto g^{-1}$, $\forall g\in G$, is a graph automorphism of X, but $\alpha\notin \mathrm{Aut}(G)$. It follows that X is nonnormal for G by Proposition 1.5.

The following theorem answered the above question completely. (See [35] for the long proof of this theorem.)

Theorem 2.11. (1) *Every finite group has at least one normal Cayley digraph*;

(2) *Every finite group other than* $Z_4\times Z_2$ *and* $Q_8\times Z_2^m$, $m\geqslant 0$, *has at least one normal Cayley graph.*

We guess that almost all Cayley digraphs and graphs are normal. To speak precisely, we propose the following conjecture.

Conjecture 1. For any positive integer n, we let $\mathscr{F}_n$ denote the class of all groups of order n, and let

$$f(n)=\min_{G\in\mathscr{F}_n}\frac{\#\text{ of normal Cayley digraphs of } G}{\#\text{ of Cayley digraphs of } G},$$

and

$$\bar{f}(n)=\min_{G\in\mathscr{F}_n,\,G\neq Q_8\times Z_2^m}\frac{\#\text{ of normal Cayley graphs of } G}{\#\text{ of Cayley graphs of } G}.$$

We conjecture that

$$\lim_{n\to\infty} f(n)=1 \quad\text{and}\quad \lim_{n\to\infty}\bar{f}(n)=1.$$

In [8] Godsil proposed a similar conjecture for GRRs and DRRs; his conjecture is stronger than the above one.

Next, we shall talk about the normality of Cayley digraphs for special groups. Example 2.2 shows that for cyclic groups of prime order we know completely which Cayley digraphs of them are normal and which are nonnormal. Unfortunately, these are the only groups for which complete information about the normality of Cayley digraphs is available. However, if we consider only edge-transitive Cayley graphs, we have the following results.

Theorem 2.12. (Alspach [2], Cheng and Oxley [10], Praeger [32,33], Wang and Xu [36]). *Let p and q be two primes with $q<p$. Then*
(1) *All vertex-primitive Cayley graphs of order pq are nonnormal.*
(2) *All disconnected Cayley graphs of order pq are nonnormal.*
(3) *The only edge-transitive connected nonnormal Cayley graphs of order pq, which have an imprimitive full automorphism group, are the following:* (i) *the lexicographic products $G(p,r)[qK_1]$, $G(q,s)[pK_1]$ and $K_2[pK_1]$, where $G(p,r)$ is the unique arc-transitive graph of order p and valency r defined by Chao* [9] *for any even divisor r of $p-1$, and $G(q,s)$ has the same definition as $G(p,r)$ for any even divisor s of $q-1$,* (ii) *the deleted lexicographic product $G(p,r)[qK_1]-qG(p,r)$ for any q and $r=p-1\geqslant 4$, or for $q>3$, and the deleted lexicographic product $G(q,s)[pK_1]-pG(q,s)$ for $q\geqslant 3$, and $K_2[pK_1]-pK_2$,* (iii) *the graphs described in Examples* 2.8 *and* 2.9 *for $q=2$.*

(In this theorem, the results for $q=2$ are extracted from [10], the results for $q=3$ are extracted from [2,36], and for $q>3$, from [32–34].)

Problem 2. *Determine all imprimitive nonnormal Cayley graphs of order pq, not only for the edge-transitive case. Do the same thing for Cayley digraphs of order pq.*

Theorem 2.13 (Li et al. [27]). *Let p be an odd prime. Then*
(1) *the only nonnormal vertex-primitive Cayley graphs of order p^2 are the graphs defined in Example* 2.7;
(2) *the only edge-transitive nonnormal Cayley graphs which have an imprimitive full automorphism group are $G(p,r)[pK_1]$ and $G(p,r)[pK_1]-pG(p,r)$, where p and r are the same as in Theorem* 2.12.

Problem 3. *Determine all imprimitive nonnormal Cayley graphs of order p^2, not only for the edge-transitive case. Do the same thing for Cayley digraphs of order p^2.*

Next, we consider the normality of Cayley digraphs of small valency. For Cayley graphs of abelian groups we have the following theorem.

Theorem 2.14 (Baik et al. [8]). *Let $X=\mathrm{Cay}(G,S)$ be a connected Cayley graph of G of valency at most* 4, *and let G be an abelian group. Then X is normal for G*

unless one of the following happens:
(1) $G = Z_4$, $S = G\backslash\{1\}$, $X = K_4$.
(2) $G = Z_4 \times Z_2 = \langle a\rangle \times \langle b\rangle$, $S = \{a, a^{-1}, b\}$, $X = Q_3$, *the cube.*
(3) $G = Z_6 = \langle a\rangle$, $S = \{a, a^3, a^5\}$, $X = K_{3,3}$.
(4) $G = Z_2^3 = \langle u\rangle \times \langle v\rangle \times \langle w\rangle$, $S = \{w, wu, wv, wuv\}$, $X = K_{4,4}$.
(5) $G = Z_4 \times Z_2 = \langle a\rangle \times \langle b\rangle$, $S = \{a, a^2, a^3, b\}$, $X = Q_3^c$, *the complement of the cube.*
(6) $G = Z_4 \times Z_2 = \langle a\rangle \times \langle b\rangle$, $S = \{a, a^{-1}, a^2b, b\}$, $X = K_{4,4}$.
(7) $G = Z_4 \times Z_2^2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $S = \{a, a^{-1}, b, c\}$, $X = C_4 \times C_4 = Q_4$, *the four-dimensional cube.*
(8) $G = Z_5$, $S = G\backslash\{1\}$, $X = K_5$.
(9) $G = Z_4 \times Z_4 = \langle a\rangle \times \langle b\rangle$, $S = \{a, a^{-1}, b, b^{-1}\}$, $X = C_4 \times C_4 = Q_4$.
(10) $G = Z_{10} = \langle a\rangle$, $S = \{a, a^3, a^5, a^7\}$, $X = K_{5,5} - 5K_2$.
(11) $G = Z_6 \times Z_2 = \langle a\rangle \times \langle b\rangle$, $S = \{a, a^{-1}, a^3, b\}$, $X = K_{3,3} \times K_2$.
(12) $G = Z_m \times Z_2 = \langle a\rangle \times \langle b\rangle$, $S = \{a, ab, a^{-1}, a^{-1}b\}$, $X = C_m[2K_1]$.
(13) $G = Z_{4m} = \langle a\rangle$, $m \geqslant 2$, $S = \{a, a^{2m+1}, a^{-1}, a^{2m-1}\}$, $X = C_{2m}[2K_1]$.

This theorem was proved independently in [8,21]. In the proof of this theorem in [8], the following lemma is basic; this lemma is a sufficient condition for the normality of Cayley digraphs of an abelian group.

Lemma 2.15. *Let $X = \mathrm{Cay}(G, S)$ be a connected Cayley digraph of an abelian group G. Assume that S satisfies the following conditions*:
(1) $S \cap S^2 = \emptyset$,
(2) *If $x, y, z, u \in S$ with $1 \neq xy = zu$, then $\{x, y\} = \{z, u\}$.*
Then X is normal for G.

Theorem 2.14 has the following corollaries.

Corollary 2.16. (1) *The only nonnormal connected Cayley graph of valency* 4 *of abelian groups of odd order is K_5 for the group Z_5.*

(2) *All connected Cayley graphs of a finite cyclic group of valency at most* 4 *are normal, except for $G = Z_4$ and $X = K_4$, or $G = Z_5$ and $X = K_5$, or $G = Z_{2m}$, $m \geqslant 3$ and $X = C_m[2K_1]$.*

These corollaries lead us to propose a problem and to make a conjecture.

Problem 4. *Determine all nonnormal connected Cayley graphs of valency* 4 *of a finite group of odd order. Are there only a finite number of such graphs?*

The known examples are K_5 for the group $G = Z_5$, and three vertex-primitive graphs with full automorphism groups PSL(2,7), PGL(2,11) and PSL(2,23) for the regular Frobenius subgroups of order 21, 55 and 253, respectively. (See [36, Example 2.3] for the first graph, and see [33, Lemma 4.3] for the last two.) Note that all these examples are vertex-primitive Cayley graphs.

Recently, however, Feng [15] found an interesting example which has an imprimitive full automorphism group.

Example 2.17. Let $G=\langle a,b,c \mid a^9=b^3=c^3=1, [a,b]=a^3, [a,c]=b, [b,c]=1\rangle$ and $S=\{a, ac\}$. Then $X=\mathrm{Cay}(G, S\cup S^{-1})$ is a nonnormal Cayley graph of valency 4 of the 3-group G of order 81, and $\mathrm{Aut}(X)$ has order $16\cdot 81$.

Conjecture 5. All connected Cayley graphs X of a finite cyclic group are normal, unless X is a complete graph, or a lexicographic product of two smaller graphs, or a deleted lexicographic product of two smaller graphs.

As a first step in building such a classification, the two-arc-transitive circulants were classified in [1]. If this conjecture is true, then it would be easy to classify all arc-transitive circulants. In fact, the problem of classifying arc-transitive circulants was part of the motivation of studying the normality of Cayley graphs.

3. Isomorphisms of Cayley digraphs

In this section we shall discuss the problem posed in Section 1 about the isomorphisms of Cayley digraphs. The problem is the following.

Problem 6. *Let G be a finite group and S a minimal generating set of G.*
(1) *Are S and $S\cup S^{-1}$ CI-subsets?*
(2) *Are the corresponding Cayley digraph and graph normal?*

The study of this problem is still in progress. I only want to mention the following results.

Proposition 3.1 (Huang and Meng [22–24]). *Let G be a finite cyclic group and S a minimal generating set of G. Let $X=\mathrm{Cay}(G,S)$ and $\overline{X}=\mathrm{Cay}(G,S\cup S^{-1})$. Let σ be an automorphism of G such that $g^\sigma=g^{-1}$, $\forall g\in G$, and let $\Sigma=\langle\sigma\rangle$. Then* $\mathrm{Aut}(X)=R(G)$ *and* $\mathrm{Aut}(\overline{X})=R(G)\Sigma$. *The answers to both questions* (1) *and* (2) *in Problem* 6 *are positive.*

For abelian groups, Li [25] gave an example which shows that the answer to question (1) is negative in general; however, if the group has odd order, then the answer to (1) is positive. Namely, he proved

Proposition 3.2. (1) *Let $G=\langle a\rangle\times\langle x\rangle\times\langle e\rangle\cong Z_3\times Z_4\times Z_2$ and let $S=\{x, xe, ax^2\}$ and $T=\{x, xe, ax^2e\}$. Then S is a minimal generating subset of G and the Cayley* *digraph* Cay(G,S) *is isomorphic to* Cay(G,T). *However, there is no automorphism of* G *which maps* S *to* T. *In other words,* S *is not a CI-subset.*

(2) *Every minimal generating subset of an abelian group of odd order is a CI-subset.*

Feng et al. [16] checked the example given in Proposition 3.2(1) and they found that the generating subset T is not minimal. They proved the following.

Proposition 3.3. *Let* G *be a finite abelian group and let* S *and* T *be two minimal generating sets of* G. *Let* $X = \mathrm{Cay}(G,S)$ *and* $Y = \mathrm{Cay}(G,T)$ *be isomorphic. Then there exists an* $\alpha \in \mathrm{Aut}(G)$ *such that* $S^{\alpha} = T$.

As a consequence of this, all generating sets with the minimum number of generators are CI-subsets. So the answer to question (1) for minimum generating sets of abelian groups is positive.

For p-groups, we have

Proposition 3.4. [Babai [3]] *Let* G *be a finite* p*-group and let* S *be a generating set of* G *with* $|S| < p$. *Then* S *is a CI-subset.*

As a consequence of this result, every minimal generating subset of a p*-group* G *is a CI-subset, if* G *has at most* $p-1$ *generators. (Note that for* p*-groups, every minimal generating subset has the same number of generators.)*

Li [25] generalized Babai's result using a deep result in group theory obtained by Gross [20]. He proved

Proposition 3.5. *Let* G *be a finite group and* p *the least divisor of* $|G|$. *Let* S *be a generating set of* G *with* $|S| < p$. *Then* S *is a CI-subset.*

As a corollary, if G *has odd order, then every generating set with two generators is a CI-subset.*

Now, we want to emphasise the two generator case, and we ask

Question 7. *Let* G *be a finite two generator group and* S *a generating set of* G *with two generators. Is* S *a CI-subset?*

To answer this question, by Proposition 3.5, we only need to consider groups of even order. Li [26] checked some 2-generating sets for simple groups. As a by-product, he proved that every 2-generating set of A_5 is a CI-subset, and the corresponding Cayley digraph is normal.

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